International Journal of Modern Physics D
Vol. 27 (2018) 1850002 (17 pages)

DOI: 10.1142/S0218271818500025

Dynamics of $f(R)$ gravity models and asymmetry of time

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Received 6 July 2017
Revised 15 September 2017
Accepted 18 September 2017
Published 11 October 2017

We solve the field equations of modified gravity for $f(R)$ model in metric formalism. Further, we obtain the fixed points of the dynamical system in phase-space analysis of $f(R)$ models, both with and without the effects of radiation. The stability of these points is studied against the perturbations in a smooth spatial background by applying the conditions on the eigenvalues of the matrix obtained in the linearized first-order differential equations. Following this, these fixed points are used for analyzing the dynamics of the system during the radiation, matter and acceleration-dominated phases of the universe. Certain linear and quadratic forms of $f(R)$ are determined from the geometrical and physical considerations and the behavior of the scale factor is found for those forms. Further, we also determine the Hubble parameter $H(t)$, the Ricci scalar R and the scale factor $a(t)$ for these cosmic phases. We show the emergence of an asymmetry of time from the dynamics of the scalar field exclusively owing to the $f(R)$ gravity in the Einstein frame that may lead to an arrow of time at a classical level.

Keywords: Modified gravity; dark energy; cosmological constant.

PACS Number(s): 98.80.−k, 95.36.+x, 04.50.−h

1. Introduction

The present state of the universe has been found to be in the phase of accelerated expansion.[1] There are several observational evidences on geometry and growth of structures such as the Supernovae Ia, Baryon Acoustic Oscillations (BAO), Cosmic Microwave Background anisotropies, weak gravitational lensing, etc.[1–4] which indicate the presence of a hitherto unknown dark energy. By all reckoning, the explanation for present accelerated expansion of the universe is a major challenge in cosmology, even though there are many approaches to explain its dynamics. The simplest candidate for dark energy is the cosmological constant with a constant equation-of-state ($w = -1$).[5] However, there are two main difficulties associated with the cosmological constant: (i) the fine tuning problem and (ii) the coincidence problem. Besides, there exist two basic approaches attempting to explain dark energy. The first approach is based on modified matter models. In this approach, $T_{\mu\nu}$ in the Einstein equations must include an exotic matter component like quintessence, k-essence, phantom energy, etc.[6–9] that takes the form of dark energy and so the responsibility of causing acceleration. The second approach is through the so-called modified gravity models wherein the late-time accelerated cosmic expansion is realized without requiring the explicit dark energy component in the universe. In these models, we have a wide range of $f(R)$ gravity models,[10] scalar–tensor theories, Gauss–Bonnet models, braneworld models, etc.[11–13] Specifically, in this paper we present an analysis of $f(R)$ models, where one modifies the laws of gravity by replacing the scalar curvature R of the Hilbert's action, or $R - 2\Lambda$, as one includes in the standard ΛCDM approach (with Λ as the cosmological constant), by an arbitrary function of R in the curvature part of the Lagrangian density. At present, there is no specific, known functional form of $f(R)$ which may satisfy all the observational conditions of cosmological viability ranging from the radiation-dominated matter to the ongoing accelerated phase. Therefore, we study the stability conditions for the respective eras and determine the corresponding forms of $f(R)$. By solving the field equations for different forms of $f(R)$, the scale factor of expansion is thus determined. From here, we find the scalar curvature R and compare them in different eras. This can lead to the determination of a time-ordering of various epochs, dominated by radiation, matter and dark energy (as a modification of gravity), respectively, throughout the evolution of the universe.

In Sec. 2, the fixed points of the dynamical system are determined within the framework of $f(R)$ models in metric formalism. To study the exclusive effects of radiation, in Secs. 3 and 4 the properties and stability of the fixed points of the dynamical system are found without and with radiation, respectively. Section 5 comprises of analysis of the behavior of $f(R)$, scale factor $a(t)$, Hubble parameter $H(t)$ (under various conditions) and scalar curvature R in radiation-dominated phase. In the following Secs. 6 and 7, we attempt to determine the form of $f(R)$, scale factor $a(t)$ and scalar curvature R for matter-dominated and the present accelerated expansion-dominated phases, respectively. Together, the time-ordering may be used further to determine an arrow of time through the cosmic evolution in Sec. 8. Finally, we conclude our results in Sec. 9.

2. Field Equations and Phase-Space Dynamics

In $f(R)$ gravity, we obtain the field equations in metric formalism, where the variation of the action is taken with respect to $g_{\mu\nu}$ related to the connections $\Gamma^{\alpha}_{\beta\gamma}$ in the usual sense (unlike the Palatini formalism where they are treated as mutually independent). We consider the field equations in the background of spatially flat

Friedmann–Lemaitre–Robertson–Walker (FLRW) spacetime with a metric

$$ds^2 = -dt^2 + a^2(t)[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)], \tag{1}$$

where $a(t)$ is time-dependent scale factor and the speed of light $c = 1$. Correspondingly, the Ricci scalar R is given by

$$R = 6(2H^2 + \dot{H}), \tag{2}$$

where $H(= \dot{a}/a)$ is the Hubble parameter and an overdot represents the derivative with respect to time. The total action using an arbitrary function $f(R)$ in the Jordan frame is given by

$$\mathcal{A} = \frac{1}{2\kappa^2}\int \sqrt{-g} f(R) d^4x + \mathcal{A}_m, \tag{3}$$

where $\mathcal{A}_m$ is the action for relativistic and nonrelativistic matter, $\kappa^2 = 8\pi G$ and g is the determinant of the metric tensor $g_{\mu\nu}$. Varying the action (3) with respect to $g_{\mu\nu}$, the field equations obtained are

$$\begin{aligned} &F(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu}, \\ &-\nabla_\mu \nabla \nu F(R) + g_{\mu\nu}\Box F(R) = \kappa^2 T_{\mu\nu}, \end{aligned} \tag{4}$$

where $F(R) \equiv \frac{\partial f}{\partial R}$ and $T_{\mu\nu}$ is the energy–momentum tensor for matter. From the above Eq. (4) and its trace, we arrive at the following expressions:

$$3FH^2 = \kappa^2(\rho_m + \rho_r) + \frac{(FR - f)}{2} - 3H\dot{F}, \tag{5}$$

$$-2F\dot{H} = \kappa^2\left(\rho_m + \frac{4}{3}\rho_r\right) + \ddot{F} - H\dot{F}, \tag{6}$$

where ρ_m and ρ_r are the energy densities of matter and radiation, respectively. The conservation equations of nonrelativistic matter and radiation are given by

$$\begin{aligned} \dot{\rho_m} + 3H\rho_m = 0; \\ \dot{\rho_r} + 4H\rho_r = 0, \end{aligned} \tag{7}$$

respectively.

Now, Eq. (5) can also be written as

$$1 = \frac{\kappa^2\rho_m}{3FH^2} + \frac{\kappa^2\rho_r}{3FH^2} + \frac{R}{6H^2} - \frac{f}{6FH^2} - \frac{\dot{F}}{FH}. \tag{8}$$

The density parameters, Ω_m, Ω_r and $\Omega_{\rm DE}$, of matter, radiation and dark energy, respectively, are defined by

$$\begin{aligned} &\Omega_m \equiv \frac{\kappa^2\rho_m}{3FH^2} = 1 - x_1 - x_2 - x_3 - x_4; \\ &\Omega_r \equiv \frac{\kappa^2\rho_r}{3FH^2} = x_4; \quad \Omega_{\rm DE} = x_1 + x_2 + x_3, \end{aligned} \tag{9}$$

with four (dimensionless) variables defined as

$$x_1 \equiv -\frac{\dot{F}}{FH}, \tag{10}$$

$$x_2 \equiv -\frac{f}{6FH^2}, \tag{11}$$

$$x_3 \equiv \frac{R}{6H^2}, \tag{12}$$

$$x_4 \equiv \frac{\kappa^2 \rho_r}{3FH^2}. \tag{13}$$

The effective equation-of-state for this system is defined by

$$w_{\text{eff}} = -1 - \frac{2}{3}\frac{\dot{H}}{H^2} = \frac{1}{3}(1 - 2x_3). \tag{14}$$

Differentiation of these variable Eqs. (10)–(13) with respect to $N = \ln a(t)$ along with the conservation Eq. (27) of nonrelativistic matter and radiation, the expressions of field Eq. (5), (6) and the Ricci scalar, together provide

$$\frac{dx_1}{dN} = -1 - x_3 - 3x_2 + x_1^2 - x_1x_3 + x_4, \tag{15}$$

$$\frac{dx_2}{dN} = \frac{x_1x_3}{m} - x_2(2x_3 - 4 - x_1), \tag{16}$$

$$\frac{dx_3}{dN} = -\frac{x_1x_3}{m} - 2x_3(x_3 - 2), \tag{17}$$

$$\frac{dx_4}{dN} = -2x_3x_4 + x_1x_4, \tag{18}$$

where

$$m \equiv \frac{d\log F}{d\log R} = \frac{Rf_{,RR}}{f_{,R}}, \tag{19}$$

with terms $f_{,R} \equiv \frac{df}{dR}$ and $f_{,RR} \equiv \frac{d^2 f}{dR^2}$.

We may define a quantity given by

$$q \equiv -\frac{d\log f}{d\log R} = -\frac{Rf_{,R}}{f} = \frac{x_3}{x_2}. \tag{20}$$

From Eq. (20), it is clear that R can be expressed as a function of $\frac{x_3}{x_2}$ and since m is a function of R, therefore m can be written in terms of q.

The fixed points of the system are obtained by equating the equations (15)–(18) to zero. Thus, the points are given by

$$\begin{aligned} &P_1 : (x_1, x_2, x_3, x_4) = (0, -1, 2, 0);\\ &\qquad\qquad \Omega_m = 0; \quad w_{\text{eff}} = -1, \end{aligned} \tag{21}$$

$$\begin{aligned}
&P_2 : (x_1, x_2, x_3, x_4) = (-1, 0, 0, 0);\\
&\qquad \Omega_m = 2; \quad w_{\text{eff}} = \frac{1}{3},
\end{aligned} \tag{22}$$

$$\begin{aligned}
&P_3 : (x_1, x_2, x_3, x_4) = (1, 0, 0, 0);\\
&\qquad \Omega_m = 0; \quad w_{\text{eff}} = \frac{1}{3},
\end{aligned} \tag{23}$$

$$\begin{aligned}
&P_4 : (x_1, x_2, x_3, x_4) = (-4, 5, 0, 0);\\
&\qquad \Omega_m = 0; \quad w_{\text{eff}} = \frac{1}{3},
\end{aligned} \tag{24}$$

$$\begin{aligned}
&P_5 : (x_1, x_2, x_3, x_4) = \left(\frac{3m}{1+m}, -\frac{1+4m}{2(1+m)^2}, \frac{1+4m}{2(1+m)}, 0\right),\\
&\qquad \Omega_m = 1 - \frac{m(7+10m)}{2(1+m)^2};\\
&\qquad w_{\text{eff}} = -\frac{m}{(1+m)},
\end{aligned} \tag{25}$$

$$\begin{aligned}
&P_6 : (x_1, x_2, x_3, x_4) = \left(\frac{2(1-m)}{1+2m}, \frac{1-4m}{m(1+2m)}, -\frac{(1-4m)(1+m)}{m(1+2m)}, 0\right);\\
&\qquad \Omega_m = 0; \quad w_{\text{eff}} = \frac{2-5m-6m^2}{3m(1+2m)},
\end{aligned} \tag{26}$$

$$\begin{aligned}
&P_7 : (x_1, x_2, x_3, x_4) = (0, 0, 0, 1);\\
&\qquad \Omega_m = 0; \quad w_{\text{eff}} = \frac{1}{3},
\end{aligned} \tag{27}$$

$$\begin{aligned}
&P_8 : (x_1, x_2, x_3, x_4) = \left(\frac{4m}{1+m}, -\frac{2m}{(1+m)^2}, \frac{2m}{1+m}, \frac{1-2m-5m^2}{(1+m)^2}\right);\\
&\qquad \Omega_m = 0; \quad w_{\text{eff}} = \frac{1-3m}{3+3m}.
\end{aligned} \tag{28}$$

3. Fixed Points without Radiation ($x_4 = 0$)

First, we consider the properties and stability of these fixed points in the absence of radiation. For stability about the fixed points (x_1, x_2, x_3) we invoke time-dependent linear perturbations $\delta x_i (i = 1, 2, 3)$ around the points in a smooth spatial background. Linearization of the equations (15)–(17) gives the first-order differential equations

$$\frac{d}{dN}\begin{pmatrix}\delta x_1\\ \delta x_2\\ \delta x_3\end{pmatrix} = M \begin{pmatrix}\delta x_1\\ \delta x_2\\ \delta x_3\end{pmatrix}, \tag{29}$$

where M is a 3×3 matrix whose components depend upon x_1, x_2 and x_3. Stability of each fixed point depends upon the eigenvalues of the matrix M obtained by taking linear perturbations around that specific point. In the absence of radiation, we have only six fixed points $P_1 - P_6$ as discussed below.

(1) Point $P_1 : (0,-1,2)$ corresponds to de-Sitter point. Here, $w_{\text{eff}} = -1$ and eigenvalues corresponding to this point are

$$-3, -\frac{3}{2} \pm \frac{\sqrt{25 - \dfrac{16}{m}}}{2}. \tag{30}$$

P_1 is stable when real parts of all the eigenvalues are negative. Hence, condition for stability is $0 < m(q = -2) < 1$, otherwise it is a saddle point. So this point can be taken as an acceleration point.

(2) Point $P_2 : (-1,0,0)$ is denoted by ϕ-matter-dominated (ϕ MDE) epoch. The eigenvalues of the 3×3 matrix of perturbations about P_2 are given by

$$\begin{aligned}-2, \frac{1}{2}\Bigg[&7 + \frac{1}{m} - \frac{m'}{m^2}q(1+q)\\ &\mp\sqrt{\left(7 + \frac{1}{m} - \frac{m'}{m^2}q(1+q)\right)^2 - 4\left(12 + \frac{3}{m} - \frac{m'}{m^2}q(3+4q)\right)}\,\Bigg],\end{aligned} \tag{31}$$

where m' is derivative of m with respect to q. If m is constant, then eigenvalues are $-2, 3, 4 + \frac{1}{m}$. In this case, P_2 is a saddle point because eigenvalues are negative and positive.

P_2 cannot be a matter-dominated point because $\Omega_m = 2$ and $w_{\text{eff}} = \frac{1}{3}$.

(3) Point $P_3 : (1,0,0)$ is the kinetic point. The eigenvalues corresponding to this point are

$$\begin{aligned}2, \frac{1}{2}\Bigg[&9 + \frac{1}{m} - \frac{m'}{m^2}q(1+q)\\ &\mp\sqrt{\left(9 - \frac{1}{m} + \frac{m'}{m^2}q(1+q)\right)^2 - 4\left(20 - \frac{5}{m} - \frac{m'}{m^2}q(5+4q)\right)}\,\Bigg].\end{aligned} \tag{32}$$

If m is constant, the eigenvalues are $2, 5, 4 - \frac{1}{m}$. In this case, P_3 is unstable for $m < 0$ and $m > \frac{1}{4}$ and a saddle otherwise.

(4) Point $P_4 : (-4,5,0)$ has eigenvalues:

$$-5, -3, 4\left(1 + \frac{1}{m}\right). \tag{33}$$

It is stable for $-1 < m < 0$ and saddle otherwise. This point cannot be used as a radiation or a matter-dominated point.

(5) Point $P_5 : \left(\frac{3m}{1+m}, -\frac{1+4m}{2(1+m)^2}, \frac{1+4m}{2(1+m)}\right)$ can be regarded as a standard matter point in the limit $m \to 0$. In this limit, $\Omega_m = 1$ and $a \propto t^{\frac{2}{3}}$. Hence, the necessary condition for this point to be a standard matter point is

$$m(q = -1) = 0. \tag{34}$$

The eigenvalues corresponding to point P_5 are given by

$$3(1 + m'), \quad \frac{-3m \pm \sqrt{m(256m^3 + 160m^2 - 31m - 16)}}{4m(m + 1)}. \tag{35}$$

For a cosmologically viable trajectory, we want a saddle matter point. Hence, the condition for a saddle matter epoch is given by

$$\begin{aligned} m(q \leq -1) &> 0, \\ m'(q \leq -1) &> -1, \\ m(q = -1) &= 0. \end{aligned} \tag{36}$$

(6) Point $P_6 : \left(\frac{2(1-m)}{1+2m}, \frac{1-4m}{m(1+2m)}, -\frac{(1-4m)(1+m)}{m(1+2m)}\right)$ can also be an acceleration-dominated point. The eigenvalues corresponding to this point are

$$-4 + \frac{1}{m}, \quad \frac{2 - 3m - 8m^2}{m(1 + 2m)}, \quad -\frac{2(m^2 - 1)(1 + m')}{m(1 + 2m)}. \tag{37}$$

Stability of this point depends on both m and m'. The condition of acceleration ($w_{\text{eff}} < -\frac{1}{3}$) depends on the value of m.

4. Fixed Points with Radiation ($x_4 \neq 0$)

Next, we include the radiation with other components of universe as a realistic case for our further study. In this case, we have eight fixed points. Stability about the fixed points (x_1, x_2, x_3, x_4) is determined in the same way as in absence of radiation. Here, we have 4×4 matrix of linear perturbations about each fixed point and four eigenvalues.

(1) Point P_1 corresponds to de-Sitter point. Here, $w_{\text{eff}} = -1$ and eigenvalues corresponding to this point are

$$-4, -3, -\frac{3}{2} \pm \frac{\sqrt{25 - \dfrac{16}{m}}}{2}. \tag{38}$$

In the presence of radiation, we have an eigenvalue -4 in addition to those in the absence of radiation. Since this eigenvalue is negative, therefore the condition of stability is the same in both cases. P_1 is stable when $0 < m(q = -2) < 1$. This point may be taken as an acceleration point. The condition of stability for this point is same as in the case of without radiation because here we have only an extra eigenvalue -4, which is negative.

(2) Point P_2 is denoted by ϕ-matter-dominated (ϕ MDE) epoch. The eigenvalues corresponding to this point are given by

$$-2, -1, \frac{1}{2}\left[7 + \frac{1}{m} - \frac{m'}{m^2}q(1+q) \right. \\ \left. \mp\sqrt{\left(7 + \frac{1}{m} - \frac{m'}{m^2}q(1+q)\right)^2 - 4\left(12 + \frac{3}{m} - \frac{m'}{m^2}q(3+4q)\right)}\right]. \tag{39}$$

P_2 is either saddle or stable point. In this case, P_2 can not be a matter point because $\Omega_m = 2$ and $w_{\text{eff}} = \frac{1}{3}$.

(3) Point P_3 is known as kinetic point. The eigenvalues for the 4×4 matrix of perturbations about this point are

$$1, 2, \frac{1}{2}\left[9 + \frac{1}{m} - \frac{m'}{m^2}q(1+q) \right. \\ \left. \mp\sqrt{\left(9 - \frac{1}{m} + \frac{m'}{m^2}q(1+q)\right)^2 - 4\left(20 - \frac{5}{m} - \frac{m'}{m^2}q(5+4q)\right)}\right]. \tag{40}$$

If m is constant, the eigenvalues corresponding to this point are $2, 5, 4 - \frac{1}{m}$. In this case, P_3 is unstable for $m < 0$ and $m > \frac{1}{4}$ and a saddle, otherwise.

(4) Point P_4 has eigenvalues

$$-5, -4, -3, 4\left(1 + \frac{1}{m}\right). \tag{41}$$

It is stable for $-1 < m < 0$ and saddle, otherwise. This point cannot be used as a radiation or a matter-dominated point.

(5) Point P_5 can be regarded as a standard matter point in the limit $m \to 0$. The eigenvalues for point P_5 are given by

$$-1, 3(1+m'), \quad \frac{-3m \pm \sqrt{m(256m^3 + 160m^2 - 31m - 16)}}{4m(m+1)}, \tag{42}$$

where m' is derivative of m with respect to q. For a cosmologically viable trajectory, we want a saddle matter point. The condition for a saddle matter epoch is given by

$$\begin{aligned} &m(q \leq -1) > 0, \\ &m'(q \leq -1) > -1, \\ &m(q = -1) = 0. \end{aligned} \tag{43}$$

(6) Point P_6 can also be an acceleration-dominated point. The eigenvalues corresponding to this point are given by

$$-\frac{2(-1+2m+5m^2)}{m(1+2m)}, \quad -4+\frac{1}{m}, \quad \frac{2-3m-8m^2}{m(1+2m)}, \quad -\frac{2(m^2-1)(1+m')}{m(1+2m)}. \tag{44}$$

The stability of this point depends on both m and m'. Condition of acceleration ($w_{\text{eff}} < -\frac{1}{3}$) depends on the value of m.

(7) Point P_7 corresponds to a standard radiation point. The eigenvalues of P_7 for constant m are $4, 4, 1, -1$. Thus, P_7 is a saddle point.

(8) Point P_8 also is a radiation point. In this case, dark energy is nonzero, therefore P_8 is acceptable as a radiation point. The eigenvalues of P_8 are given by

$$1, 4(1+m'), \quad \frac{m-1\pm\sqrt{81m^2+30m-15}}{2(m+1)}. \tag{45}$$

Point P_8 is a saddle point in the limit $m \to 0$. The acceptable radiation-dominated point P_8 lies at point $(0,-1)$ in the (m,q) plane.

5. Dynamics of Radiation-Dominated Phase

For radiation-dominated era, the phase-space analysis shows that we can find two points P_7 and P_8 in the limit $m \to 0$ as radiation points because for these points the density parameter Ω_r is 1 and the effective equation-of-state w_{eff} is $\frac{1}{3}$. Here, we take the point P_8 in the limit $m \to 0$ as a radiation point because in this case dark energy is nonzero. This point lies on the line $m = -q-1$ in the (m,q) plane. Hence, the necessary condition for this point to exist as an exact standard radiation point is given by

$$m(q=-1) \approx 0. \tag{46}$$

From the definition of q and the above condition, the form of $f(R)$ for radiation-dominated era is given by

$$f(R) = \alpha R, \tag{47}$$

where α is an integration constant. The standard radiation point is obtained by substitution of $m \approx 0$ in the radiation point of $m(q)$ curve. Under this condition, the effective equation-of-state is

$$w_{\text{eff}} = \frac{1}{3}. \tag{48}$$

Using Eqs. (14) and (48), the Hubble parameter is given by

$$H(t) = \frac{1}{(2t-c_1)}, \tag{49}$$

where c_1 is an integration constant.

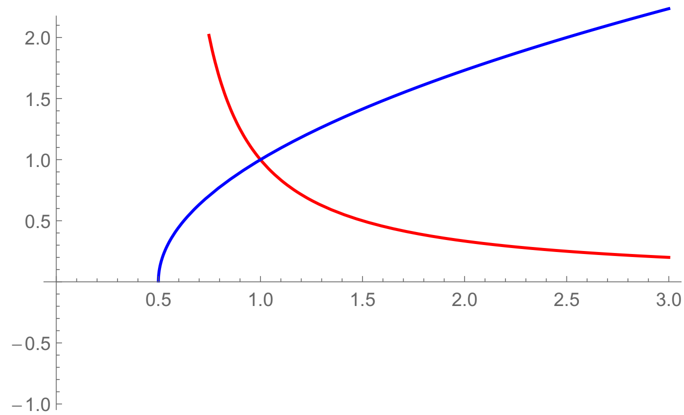

Fig. 1. (Color online) Plot for variation of the scale factor $a(t)$ (blue curve) and Hubble parameter $H(t)$ (red curve) along the y-axis with cosmic time (t) along the x-axis in radiation-dominated phase. These curves correspond to $(c_1, c_2) \equiv (1, 1)$. This solution of dynamical equations in $f(R)$ gravity is obtained using the phase-space analysis. These plots show that the behavior of the scale factor and the Hubble parameter in radiation-dominated phase of the universe in $f(R)$ gravity is similar to that of the radiation-dominated phase of the ΛCDM model.

The scale factor for this era is given by

$$a(t) = c_2(2t - c_1)^{\frac{1}{2}}, \tag{50}$$

where c_2 is another integration constant.

In radiation-dominated phase we confirm that the scale factor $a(t) \propto t^{\frac{1}{2}}$, which is same as in the case of standard model. Figure 1 shows the variation of the Hubble parameter $H(t)$ and scale factor $a(t)$ with time t in radiation phase. As expected, the Ricci scalar R for radiation-dominated era is given by

$$R = 0. \tag{51}$$

6. Dynamics of Matter-Dominated Era

From the field equations (5) and (6) we obtain the following equation:

$$-\frac{\kappa^2 \rho_r}{3} + 3FH^2 + F\dot{H} - \frac{f}{2} - 2H\dot{F} - \ddot{F} = 0. \tag{52}$$

In phase-space analysis of dynamical system, there is a point P_5 which represents a standard matter era in the limit $m \to 0$ because, in this case the density parameter of matter, $\Omega_m = 1$, and effective equation-of-state, $w_{\text{eff}} = 0$. In matter-dominated phase of the universe

$$m(q = -1) \approx 0. \tag{53}$$

Using the definition of q or m, the form of $f(R)$ is given by

$$f(R) = \beta R, \tag{54}$$

where β is an integration constant. Thus, in matter-dominated phase the form of $f(R)$ is similar as in the case of radiation-dominated phase.

In matter-dominated phase, we neglect the energy density of radiation i.e. $\rho_r = 0$. For $f(R) = \beta R$, $F = \beta$ and therefore $\dot{F} = 0$. Using Eqs. (52) and (2) and these values of F and $\dot{F}$, the time evolution of the Hubble parameter is expressed as

$$H(t) = \frac{1}{\left(\frac{3}{2}t - c_3\right)}, \tag{55}$$

where c_3 is an integration constant.

The scale factor in this phase is given by the expression

$$a(t) = c_4 \left(\frac{3}{2}t - c_3\right)^{\frac{2}{3}}. \tag{56}$$

From Eqs. (2) and (55) the Ricci scalar in matter-dominated phase is given by

$$R = \frac{3}{\left(\frac{3}{2}t - c_3\right)^2}. \tag{57}$$

The variation of Hubble parameter $H(t)$, scale factor $a(t)$ and Ricci scalar R, with time is plotted in Fig. 2. Hubble parameter $H(t)$, scale factor $a(t)$ and Ricci scalar R in this phase can also be calculated by the same procedure as we followed in the radiation era. The expressions for these parameters are the same in both approaches. For $m \approx 0$, the effective equation-of-state is given by

$$w_{\text{eff}} = 0. \tag{58}$$

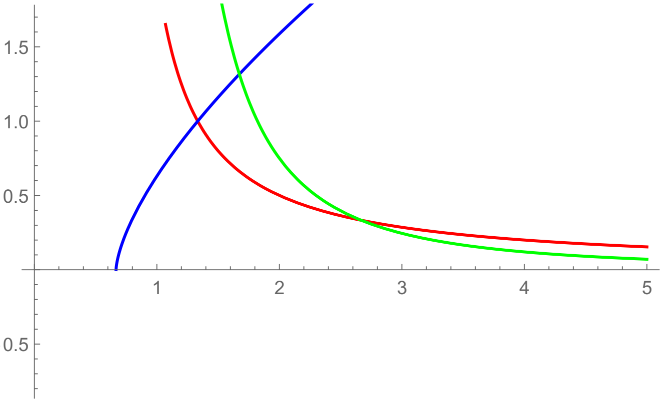

Fig. 2. (Color online) Plot for variation of scale factor $a(t)$ (blue curve), Hubble parameter $H(t)$ (red curve) and Ricci scalar R (green curve) along the y-axis with cosmic time t along the x-axis in matter-dominated phase. These curves correspond to $(c_3, c_4) \equiv (1, 1)$. These plots of different parameters in $f(R)$ gravity are similar to the plots of the same parameters in the standard model of cosmology. Here, we obtained this form of $f(R)$ using the viability conditions in the phase-space analysis.

These expressions of scale factor $a(t)$, Hubble parameter $H(t)$ and Ricci scalar R in matter-dominated phase are similar to the expressions of standard (ΛCDM) model.

7. Dynamics of Accelerated Expansion Dominated Phase

In the phase-space analysis, there is a point P_1, for which effective equation-of-state is

$$w_{\rm eff} = -1, \tag{59}$$

and the density parameter of dark energy $\Omega_{\rm DE} = 1$. Therefore, this point can be taken as the accelerated expansion point. It is called the de Sitter point. If we take the de Sitter expansion, this point is stable when $0 < m < 1$ at $q = -2$. Now, from the definition of q, the form of $f(R)$ in this phase is given by

$$f(R) = \gamma R^2. \tag{60}$$

We have the effective equation-of-state as

$$w_{\rm eff} = -1 - \frac{2}{3}\frac{\dot{H}}{H^2}. \tag{61}$$

Now, using Eqs. (59) and (61), in this phase we get the constant value of the Hubble parameter as

$$H(t) = c_5, \tag{62}$$

where c_5 is an integration constant. Therefore, the Ricci scalar in this phase is given by

$$R = 12c_5^2. \tag{63}$$

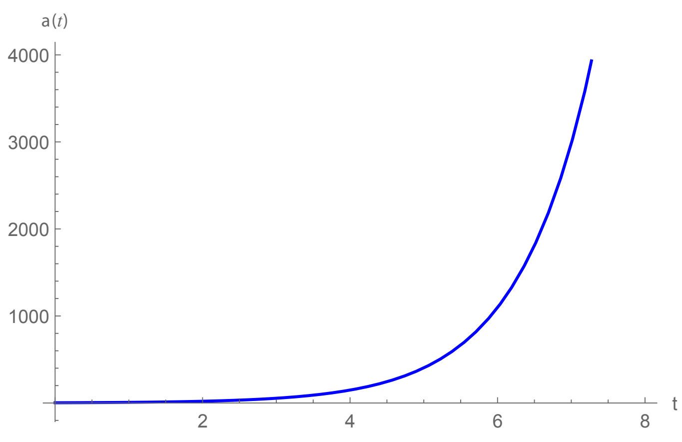

Fig. 3. Plot for variation of the scale factor $a(t)$ with cosmic time t in acceleration-dominated phase. Here, the scale factor grows exponentially with time t. It shows that the $f(R) = \gamma R^2$ models behave like the cosmological constant Λ.

Using the expression of Hubble parameter $H(t)$, the scale factor is given by

$$a(t) = e^{c_5 t + c_6}, \tag{64}$$

where c_6 is another integration constant. We can also find out these parameters using Eqs. (52) and (2) in the spatially flat universe.

Here, Fig. 3 shows the variation of scale factor $a(t)$ with time. It is clear that the expansion in this phase is exponential. This behavior is found to be similar to the case of the standard ΛCDM model.

8. Asymmetry of Time

We rewrite the action (3) in the form

$$\mathcal{A} = \int \sqrt{-g}\left(\frac{1}{2\kappa^2}FR - U\right) d^4x + \mathcal{A}_m, \tag{65}$$

where

$$U = \frac{FR - f}{2\kappa^2}. \tag{66}$$

It is possible to derive an action in the Einstein frame under the conformal transformation

$$\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}, \tag{67}$$

where Ω^2 is the conformal factor and a tilde denotes the quantities pertaining to the Einstein frame. The corresponding Ricci scalars in the two frames are related as

$$R = \Omega^2(\tilde{R} + 6\tilde{\Box}\omega - 6\tilde{g}^{\mu\nu}\partial_\mu\omega\partial_\nu\omega), \tag{68}$$

where

$$\omega \equiv \ln\Omega, \quad \partial_\mu\omega \equiv \frac{\partial\omega}{\partial\tilde{x}^\mu}, \quad \tilde{\Box}\omega \equiv \frac{1}{\sqrt{-\tilde{g}}}\partial_\mu(\sqrt{-\tilde{g}}\tilde{g}^{\mu\nu}\partial_\nu\omega). \tag{69}$$

Thus, the action (65) is transformed as

$$\mathcal{A} = \int d^4x\sqrt{-\tilde{g}}\left[\frac{1}{2\kappa^2}F\Omega^{-2}(\tilde{R} + 6\tilde{\Box}\omega - 6\tilde{g}^{\mu\nu}\partial_\mu\omega\partial_\nu\omega) - \Omega^{-4}U\right] + \mathcal{A}_m. \tag{70}$$

The linear action in $\tilde{R}$ can be written by choosing

$$\Omega^2 = F. \tag{71}$$

We consider a new scalar field ϕ defined by

$$\kappa\phi \equiv \sqrt{\frac{3}{2}}\ln F. \tag{72}$$

Using these relations, the action in Einstein frame is found as[14]

$$\mathcal{A} = \int d^4x\sqrt{-\tilde{g}}\left[\frac{1}{2\kappa^2}\tilde{R} - \frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right] + \mathcal{A}_m. \tag{73}$$

where

$$V(\phi) = \frac{U}{F^2} = \frac{FR - f}{2\kappa^2 F^2}. \tag{74}$$

On varying the action (73) with respect to ϕ in the absence of matter (relativistic and nonrelativistic, both), we get

$$\frac{d^2\phi}{d\tilde{t}^2} + 3\tilde{H}\frac{d\phi}{d\tilde{t}} + V_{,\phi} = 0, \tag{75}$$

with $V_{,\phi}$ implying the usual derivative with respect to ϕ. The energy density and pressure of the above homogeneous scalar field, respectively, are

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi); \quad p = \frac{1}{2}\dot{\phi}^2 - V(\phi), \tag{76}$$

while the scalar field equation of motion is given by (75).

In the oscillating Tolman universe models, the maximum value of the scale factor increases in the successive cycles, assuming the presence of a viscous fluid with pressure

$$p = p_0 - 3\zeta H, \tag{77}$$

where p_0 is the equilibrium pressure and ζ is the coefficient of bulk viscosity.[15] It is clear from Eq. (77) that $p < p_0$ during expansion ($H > 0$), whereas $p > p_0$ during contraction. This asymmetry during the expanding and contracting phases results in the growth of both energy and entropy. This increase in entropy makes the amplitude of successive expansion cycles larger leading to an arrow of time.

In our discussion of $f(R)$ gravity models, the term $3\tilde{H}\frac{d\phi}{d\tilde{t}}$ in (75) behaves like friction and damps the motion of the scalar field when the universe ($H > 0$). In a contracting universe, however, this term behaves like anti-friction and accelerates the motion of the scalar field. A scalar field with the potential $V = M^2\phi^2$ gives $p \simeq -\rho$ when $H > 0$ and $p \simeq \rho$ when $H < 0$. These results are in conformity with those of Tolman.

Further, we derive different potentials in all the phases of the universe. In radiation-dominated phase we have $f(R) = \alpha R$, therefore, the potential given by Eq. (74) is $V(\phi) = 0$ for this phase. Similarly, for matter-dominated phase we have $f(R) = \beta R$ and $V(\phi) = 0$. For accelerated expansion phase, the form of the Lagrangian is given by including $f(R) = \gamma R^2$ and the potential for this phase is $V(\phi) = \frac{1}{8\gamma\kappa^2}$.

The scalar field given by Eq. (72) gives $\kappa\phi = \sqrt{\frac{3}{2}}\ln\alpha$ for matter-dominated phase and $\kappa\phi = \sqrt{\frac{3}{2}}\ln\beta$ for radiation-dominated phase. For accelerated expansion phase

$$\kappa\phi = \sqrt{\frac{3}{2}}\ln(2\gamma R). \tag{78}$$

The general solution of Eq. (75) for the potential $V(\phi) = \frac{1}{2}M^2\phi^2$ is given by

$$\phi = \frac{\phi_0}{2}\exp(-\Psi\tilde{t}\,)\left(1 + \frac{\Psi}{\sqrt{\Psi^2 - M^2}}\right)\exp(\sqrt{\Psi^2 - M^2})\tilde{t} + \frac{\phi_0}{2}\exp(-\Psi\tilde{t}\,)\left(1 - \frac{\Psi}{\sqrt{\Psi^2 - M^2}}\right)\exp(-\sqrt{\Psi^2 - M^2})\tilde{t}, \tag{79}$$

where ϕ_0 is the maximum value of scalar field ϕ at $\tilde{t} = 0$ ($\tilde{t}$ being the time in the Einstein frame, henceforth), $\Psi = \frac{3\tilde{H}}{2}$ and M is the mass of the scalar field. In this solution, three cases arise depending on the value of $\sqrt{\Psi^2 - M^2}$.

(i) When $M^2 > \psi^2$. In this case, the oscillations are damped harmonic oscillations.
(ii) When $M^2 < \Psi^2$. In this case, $(\Psi^2 - M^2)$ is a positive quantity and there is an exponential term with negative power. So, the field dies off exponentially with time. There is no oscillation and the motion becomes over-damped.
(iii) When $M^2 = \Psi^2$. This is a special case, appearing as the critical damping of the scalar field. The solution is given as an aperiodic damping.

It is clear that in all the above cases we get a damping either periodic or aperiodic that owes its relation to $\Psi = \frac{3\tilde{H}}{2}$ and mass M of the scalar field ϕ. When $\tilde{H}$ is positive and constant, Ψ damps the scalar field. Figure 4 shows the behaviour of the scalar field for all three cases of the solution, when $\tilde{H}$ is a positive constant. When Ψ is negative and constant, the motion of the scalar field accelerates in all the three cases. Thus, there are two cases in which scalar field $\phi(t)$ either grows or decays depending upon the value of the Hubble parameter in the Einstein frame. This

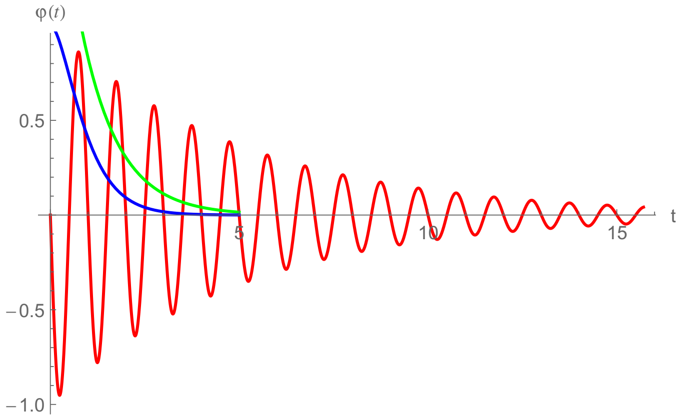

Fig. 4. (Color online) Plot for the variation of the scalar field ϕ along y-axis with time t along x-axis. Here, Red, Green and Blue curves show the behavior of scalar field $\phi(t)$ for three different cases of solution (i) $M^2 > \Psi^2$, (ii) $M^2 < \Psi^2$ and (iii) $M^2 = \Psi^2$, respectively. We have taken the positive and constant value of Ψ. If, the value of Ψ is negative and constant, then the scalar field grows. Hence, it is clear that the scalar field either accelerates or decays in all the three cases of solution. The acceleration or slow down of scalar field shows that the field is not symmetric in time. It is this asymmetry which leads to a classically observable arrow of time.

nature of the scalar field expresses an asymmetry in time. In turn, this asymmetry during the expansion $\tilde{H} > 0$ or contraction $\tilde{H} < 0$ of the universe further leads to an arrow of time. We also find that the scalar field is not symmetric under the reversal of time i.e. $(t \leftrightarrow -t)$ for positive and negative $\Psi = \frac{3\tilde{H}}{2}$. This dissipation of the scalar field indicates a crucial asymmetry in time in form of its arrow from past to future.

9. Conclusion

While describing the $f(R)$ models of modified gravity, we have studied the properties and stability of the fixed points of the dynamical system against the time-dependent perturbations in a smooth spatial background. We have discussed the role of radiation in our analysis and compared it with the case without radiation. It is found that the nature of the fixed points with radiation remains unaltered as that without radiation (except that with radiation we have the emergence of an extra eigenvalue for each point). Of course, the future discussion would bring out an analysis of the fixed points against spatial perturbations as well.

We have determined the forms of $f(R)$ for different phases of the universe, over radiation, matter and acceleration-dominated eras, by using the necessary conditions for the phase-space analysis to reach eventually at a cosmologically viable model. The scale factor $a(t)$, the Hubble parameter $H(t)$, Ricci scalar R have been determined for these phases, with a view that their ordering over the entire evolution of the universe may explain the emergence of an arrow of time, more comprehensively in a future study. While these model parameters are found to be consistent with ΛCDM model, the crucial issue is that the scalar field ϕ, that owes its origin exclusively to $f(R)$ gravity, may be invoked to explain the arrow of time based on its explicit asymmetry on a classical level. While the scalar–tensor theories may have the forms of potentials matching with $f(R)$, the results obtained on the stability conditions and their ordering extending through the overall history may not be reproduced in such theories. We would further explore this fundamental aspect of the nature of time and its observational viability within the modified gravity sector.

Acknowledgments

MMV thanks Edward Kolb at the Kavli Institute of Cosmological Physics, the University of Chicago, USA for hosting the visit and Varun Sahni at the Inter-University Centre for Astronomy and Astrophysics (IUCAA), Pune, India for discussion and facilities. Both authors thank IUCAA for the support under the associateship programme.

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