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Cosmological wheel of time: A classical perspective of $f(R)$ gravity

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It is shown that the structures in the universe can be interpreted to show a closed wheel of time, rather than a straight arrow. An analysis in $f(R)$ gravity model has been carried out to show that due to local observations, a small arc at any given spacetime point would invariably indicate an arrow of time from past to future, though on a quantum scale it is not a linear flow but a closed loop, a fact that can be examined through future observations.

Keywords: Cosmological arrow of time; modified gravity; cosmological constant.

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1. Introduction

The cosmological arrow of time has been explained in cyclic universe without any dissipation in the presence of scalar field.[1] The cosmological arrow of time may be linked to the thermodynamic arrow by the second law of thermodynamics. The time asymmetry is also associated with dissipative fluid as Tolman introduced a viscous fluid to generate an arrow of time in cyclic cosmology.[2] Eddington once related an arrow to the increase of entropy in isolated systems.[3] There is an approach related with the entropy of a system in which time asymmetry might be a feature of a subsystem to which we belong and therefore time's arrow may be perspectival.[4] It is also shown by some authors that the dark energy (positive cosmological constant) further supports the time asymmetry.[5] One of the most suitable candidates for dark energy is the cosmological constant Λ,[6] even though we do not know its precise origin, in addition to the issues related with the coincidence and fine tuning problems. The other approaches to explain the dark energy include modified gravity models[7–9] and the modified matter models.[10,11] Among the former class, $f(R)$

dark energy models are included as the simplest modified gravity models.[12–14] The time asymmetry is shown in $f(R)$ gravity using the dissipation of the scalar field.[18] In the present paper, we study the issue of the arrow of time in the $f(R)$ gravity background.

In Sec. 2, we determine the form of potential due to $f(R)$ term in the modified gravity model by using the solutions of the field equations of $f(R) = \alpha R^2$ dark energy model.[15] In Sec. 3, we show a local arrow of time by taking the perturbations in a time dependent larger mass $M(t)$. Finally, results are concluded in Sec. 4.

2. Dynamics of Effective $f(R)$ Potential

A class of $f(R)$ dark energy models attempt to explain the early inflation as well as the present cosmic accelerated expansion without using any exotic matter component.[14] For the spatially flat Friedmann–Robertson–Walker (FRW) metric given as

$$ds^2 = -dt^2 + a^2(t)[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)], \tag{1}$$

with $a(t)$ as the scale factor, the Ricci scalar is given by

$$R = 6(2H^2 + \dot{H}), \tag{2}$$

where H is the Hubble parameter and the overdot represents the derivative with respect to time. In the absence of matter (or with negligible matter), the [00] component of the dynamical equation for $f(R)$ model in the metric formalism gives

$$3FH^2 = \frac{(FR - f)}{2} - 3H\dot{F}, \tag{3}$$

where $F(R) = \frac{\partial f}{\partial R} = f'(R)$ and the overdot denotes the derivative with respect to time. Using the expression for Ricci scalar, it is possible to write

$$\frac{d}{dt} = \dot{H}\frac{d}{dH} = \left(\frac{R}{6} - 2H^2\right)\frac{d}{dH}. \tag{4}$$

Substituting Eq. (4) into Eq. (3), we get

$$3H\left(\frac{R}{6} - 2H^2\right)f''(R)\frac{dR}{dH} = \frac{(f'(R)R - f(R))}{2} - 3H^2 f'(R). \tag{5}$$

For $f(R) = \alpha R^2$ model, where α is a constant, Eq. (5) becomes

$$(R - 12H^2)\left(H\frac{dR}{dH} - \frac{R}{2}\right) = 0. \tag{6}$$

The Ricci scalar from Eq. (6) as function of Hubble parameter H is given by

$$R(H) = R_0\left(\frac{H}{H_0}\right)^{\frac{1}{2}}, \tag{7}$$

where R_0 and H_0 are constants. Using Eqs. (2) and (7), we obtain

$$\frac{da}{a} = \frac{HdH}{\dot{H}} = \frac{HdH}{\frac{R_0 H^{\frac{1}{2}}}{6(H_0)^{\frac{1}{2}}} - 2H^2}. \tag{8}$$

On integration Eq. (8) gives

$$a(t) = a_0 \left[\frac{\chi - \left(\frac{H}{H_0}\right)^{\frac{3}{2}}}{\chi - 1} \right]^{\frac{-1}{3}}, \tag{9}$$

where a_0 is a constant and $\chi = \frac{R_0}{12H_0^2}$, whereas an inversion of this expression yields

$$H = H_0 \left[\chi - (\chi - 1) \left(\frac{a}{a_0} \right)^{-3} \right]^{\frac{2}{3}}. \tag{10}$$

We consider the motion of a particle of unit mass, $m = 1$ in the gravitational field of another mass M, which is very large in comparison to the mass of the test particle. Here, we have an extra potential due to the $f(R)$ gravity background. Specifically, the contribution of $f(R) = \alpha R^2$ model to the acceleration of a test mass m in the field of a larger mass $M \gg m$ can be obtained by using Eqs. (9) and (10). From Eq. (10), we have

$$H = \frac{\dot{a}}{a} = H_0 \left[\chi - (\chi - 1) \left(\frac{a}{a_0} \right)^{-3} \right]^{\frac{2}{3}}, \tag{11}$$

which gives

$$\dot{a} = aH_0 \left[\chi - (\chi - 1) \left(\frac{a}{a_0} \right)^{-3} \right]^{\frac{2}{3}}. \tag{12}$$

Now, differentiating Eq. (12) with respect to time and using Eq. (9) for scale factor $a(t)$, we obtain the acceleration given as

$$\ddot{a} = \frac{H_0^2[\chi a^3 + (\chi - 1)a_0^3][\chi a^{\frac{3}{2}} - (\chi - 1)a_0^3 a^{\frac{-3}{2}}]^{\frac{1}{3}}}{a^{\frac{5}{2}}}, \tag{13}$$

where $\ddot{a}$ represents the second derivative of distance with respect to time t. Integrating equation (13) with respect to a we found the potential due to $f(R)$ gravity. It is given by

$$V_f(a) = -\int \ddot{a} da = -\frac{1}{2} H_0^2 \left(\frac{(a^3\chi - (\chi - 1)a_0^3)^{\frac{4}{3}}}{a^2} \right). \tag{14}$$

We can ignore the terms containing higher orders in the denominator for large a. Then, the potential $V_f(a)$ becomes

$$V_f(a) = -\frac{1}{2} H_0^2 \chi^{\frac{4}{3}} a^2. \tag{15}$$

The potential due to the mass M is

$$V_M(a) = -\frac{GM}{a}. \tag{16}$$

Here, we would have the potential due to the mass M with an additional part due to the $f(R) = \alpha R^2$ model. If l is the constant magnitude of the angular momentum of the test particle, then the total effective potential is given by

$$V(a) = \frac{l^2}{2a^2} - \frac{GM}{a} - \frac{1}{2}H_0^2\chi^{\frac{4}{3}}a^2, \tag{17}$$

whereas, the energy of the test particle of unit mass is given by

$$E = \frac{\dot{a}^2}{2} + V(a). \tag{18}$$

The energy of the test particle is conserved before the appearance of the potential term due to the $f(R)$ effects, and it would be conserved with this potential also, owing to the fact that it is a function of distance only. Thus, under the prevailing conditions, the corresponding forces behave as the conservative forces and the conservation of energy is given by

$$\frac{d}{dt}\left(\frac{1}{2}\dot{a}^2 + \frac{l^2}{2a^2} + V(a)\right) = 0. \tag{19}$$

Before the addition of potential due to $f(R)$ gravity, the total effective potential is given by

$$V(a) = \frac{l^2}{2a^2} - \frac{GM}{a}, \tag{20}$$

and after the addition of extra potential due to $f(R)$, we obtain the total potential given by Eq. (17).

The motion of the test particle is influenced by the matter mass M and the $f(R)$ gravity. The term $\frac{l^2}{2a^2}$ in Eq. (17) arises from the total angular momentum of the system. Since the problem depicts the spherically symmetry, the angular momentum vector $\mathbf{L}$ is conserved. It is always perpendicular to the radius vector $\mathbf{r}$. If $\mathbf{L} = 0$, then the motion will be along a straight line through the center of force. This is the case of central force motion. For simplicity, we can replace it by a hard wall imposed at small distance a_0.

We consider the characteristic scale of energy as $\bar{E}$ and that of distance as $\bar{a}$, and write $V(a) = \bar{E}V(r)$ in terms of the dimensionless $r = \frac{a}{\bar{a}}$ and $V(r)$. Therefore, the potential $V(a)$ can be written as

$$V(r) = -\frac{\beta}{r} - \frac{\gamma}{2}r^2, \quad r \geq r_0, \tag{21}$$

where $\beta = GM$, $\gamma = \frac{H_0^2\chi^{\frac{4}{3}}}{2}$ and $r > r_0$ is the hard wall-condition. We have

$$r_0 = \frac{a_0}{\bar{a}}. \tag{22}$$

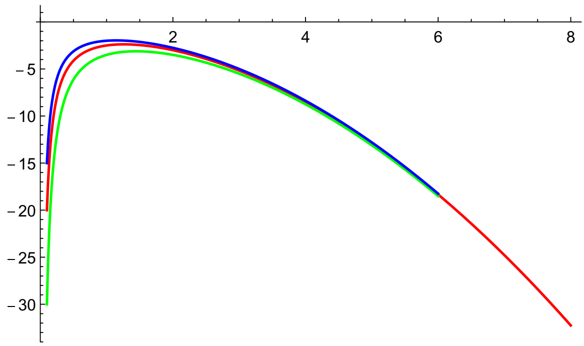

Fig. 1. (Color online) The effective potential $V(r)$ along y-axis versus distance r along x-axis. Red curve is for ($\beta = \gamma = 1$) and $r_0 = 0.10$. Blue curve stands for $\beta = 0.50$ and $\gamma = 1$ and $r_0 = 0.10$, and the green curve is plotted for $\beta = 2$ and $\gamma = 1$ and $r_0 = 0.10$.

The potential is maximal at

$$r_m = \left(\frac{\beta}{\gamma}\right)^{\frac{1}{3}} \tag{23}$$

and its maximum value is

$$V(r_m) = -\frac{3}{2}(\beta)^{\frac{2}{3}}\gamma^{\frac{1}{3}}. \tag{24}$$

The conditions for bounded and unbounded motion are given by $V(r_m) > E > V(r_0)$ and $V(r_0) < V(r_m) < E$, respectively. The maximal potential $V(r_m)$ and maximal distance r_m depend upon β and γ. The behaviour of the above effective potential is shown in Fig. 1.

3. Classical Arrow but Quantum Wheel of Time

We consider the thermodynamic equilibrium of a single particle system in statistical mechanics where the energy of the system is conserved. This is defined by the microcanonical ensemble of the system. Now the energy of the system is defined by the mass M of the massive body and it can be taken as a parameter of the system. If the mass M changes slowly, the energy of the system also varies and is described by the adiabatic invariant quantity[16]

$$I = \oint p dq, \tag{25}$$

where p is the generalized momentum and q is the generalized coordinate. We take the motion in one degree of freedom, therefore the phase-space is two-dimensional with coordinates r and p. Thus, the adiabatic invariant quantity in our case is

$$I = \iint (dp)(dr). \tag{26}$$

We have $p = \sqrt{2(E(t) - V(r))}$, therefore the adiabatic invariant becomes

$$I = \int_{r_0}^{\bar{r}} dr \sqrt{(2(\epsilon(t) - V(r, \beta(t))),} \tag{27}$$

where $\epsilon(t) = \frac{E(t)}{\bar{E}}$ and $\bar{r}$ is the radial distance for which the motion of the particle is admissible.

The logarithm of the adiabatic quantity I provides the entropy of the system. The conservation of I is related with the second law of thermodynamics. For a given $\beta(t)$ and $\gamma(t)$, $\bar{r}(t)$ is always smaller than the largest possible distance of the bounded motion:

$$\bar{r}(t) \leq r_m(t) = \left(\frac{\beta}{\gamma}\right)^{\frac{1}{3}}. \tag{28}$$

Now, we take the slow perturbation in the larger mass M. Since adiabatic invariant describes the energy, and the change in mass affects the energy of the system, so the adiabatic invariant is affected by any changes in mass. There are two approaches in which the perturbation in the mass M can be taken.

(1) When $M(t) \propto \beta(t)$ decreases slowly. During the slow decrease of $\alpha(t)$, $\epsilon(t)$ grows faster than maximal potential energy $\phi(r_m)$. Here, the change in $\alpha(t)$ is slow, otherwise, $\epsilon(t)$ will not change much and will stay bound. It is shown in Fig. 2 where the dashed curve represents the maximum value of potential energy $\phi(r_m)$ for $\alpha = \beta = \gamma = 1$ and $r_0 = 0.10$. The curves below the dashed line refers to a bounded motion. It is clear that when $\alpha(t)$ decreases slowly, this motion becomes unbounded, i.e. its energy rises above the dashed line.

(2) When $M(t) \propto \beta(t)$ increases. In this case, energy $\epsilon(t)$ and the potential $V(r_m)$, both increase when the mass increases slowly. But, if the variation in the mass is sudden, then the Hamiltonian changes while the state of the ensemble does not change. Therefore, the energy of the system does not change very much

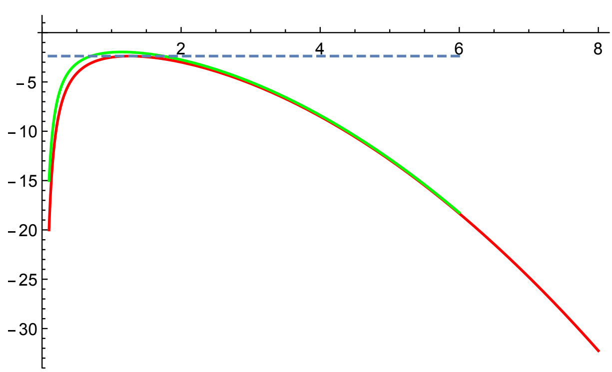

Fig. 2. (Color online) The effective potential $V(r)$ along y-axis versus distance r along x-axis. Red curve stands for ($\beta = \gamma = 1$) and $r_0 = 0.10$, while the green curve is for $\beta = 0.50$ and $\gamma = 1$ and $r_0 = 0.10$. Dashed curve is plotted for $V(r_m)$ for $\beta = \gamma = 1$ and $r_0 = 0.10$.

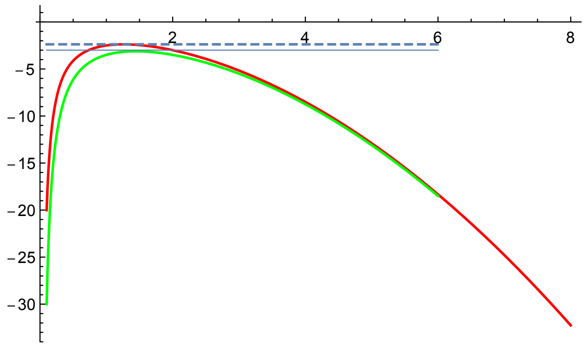

Fig. 3. (Color online) The effective potential $\phi(r)$ along y-axis versus distance r along x-axis. Red curve is plotted for $(\beta = \gamma = 1)$ and $r_0 = 0.10$. Green curve for $\beta = 2$ and $\gamma = 1$ and $r_0 = 0.10$. Dashed curve is $V(r_m) = -2.38$, $\beta = \gamma = 1$ and $r_0 = 0.10$. Thin line is an example of finite motion at energy $\epsilon = -3$. When β slowly changes from $\beta = 1$ to $\beta = 2$, the energy decreases and always refers to finite motion. If β changes sufficiently fast, the initial energy does not change much and the motion becomes unbounded.

and the system becomes unbounded,[17] it is shown in Fig. 3. Therefore, in both scenarios, the motion changes from bounded to unbounded. This is an example of irreversibility of time.

The detailed structural analysis of cosmic matter distribution shows that the maximal Lyapunov exponent is given by

$$\frac{1}{t} \ln \frac{\operatorname{mod} \delta z(t)}{\delta z_0} \to \sqrt{\lambda} \tag{29}$$

for $t \to \infty$. It determines the increasing phase-space volume over small arc of time. However, the Lyapunov spectrum over the global cosmic scales leads to a closed Hamiltonian with constant phase volume, resulting in a constant entropy. This implies the existence of a closed loop of time perhaps as a wheel, over the scales of Lyapunov time when the nearby trajectories are well resolved for the stability of the universe.

4. Conclusion

We have argued that the large scale structures with the background $f(R)$ contribution to potential must show a repulsive order causing the unbounded motion of the test particles such that the phase-space volume remains constant over those scales. We found an asymmetry resulting due to the unbound motion of the initially bound system. Owing to this fact, any local observation of structures would invariably reveal a straight arrow of time just like a tangent to a small local arc of a circle or loop. However, on the quantum scales, the future analysis is expected to show that the time axis must turn into a wheel and the usual arrow of time would have vanishing features.

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References

1. V. Sahni, Y. Shtanov and A. Toporensky, *Class. Quantum Grav.* **32** (2015) 182001.
2. R. C. Tolman, *Relativity Thermodynamic and Cosmology* (Clarendon, UK, 1934).
3. A. S. Eddington, *The Nature of the Physical World* (The Macmillan Co., New York, 1928).
4. C. Rovelli, arXiv:1505.01125v2.
5. A. E. Allahverdyan and V. G. Gurzadyan, *Phys. Rev. E* **93** (2016) 052125.
6. V. Sahni and A. A. Starobinski, *Int. J. Mod. Phys. D* **9** (2000) 373.
7. L. Amendola, D. Polarski and S. Tsujikawa, *Phys. Rev. Lett.* **98** (2007) 131302; A. A. Starobinsky, *Phys. Lett. B* **91** (1980) 99; V. Sahni and Y. Shtanov, *J. Cosmol. Astropart. Phys.* **0311** (2003) 014.
8. L. Amendola, R. Gannouji, D. Polarski and S. Tsujikawa, *Phys. Rev. D* **75** (2007) 083504.
9. S. Tsujikawa, *Lect. Notes Phys.* **800** (2010) 99.
10. R. R. Caldwell, R. Dave and P. J. Steinhardt, *Phys. Rev. Lett.* **80** (1998) 1582; T. Chiba, T. Okabe and M. Yamaguchi, *Phys. Rev. D* **62** (2000) 023511; C. Armendariz-Picon, V. F. Mukhanov and P. J. Steinhardt, *Phys. Rev. D* **63** (2001) 103510.
11. R. R. Caldwell, *Phys. Lett. B* **545** (2003) 23.
12. S. Capozziello, *Int. J. Mod. Phys. D* **11** (2002) 483.
13. S. M. Carroll, V. Duvvuri, M. Trodden and M. S. Turner, *Phys. Rev. D* **70** (2004) 043528.
14. S. Capozziello, V. F. Cardone, S. Carloni and A. Troisi, *Int. J. Mod. Phys. D* **12** (2003) 1969.
15. S. Domazet *et al.*, *Int. J. Mod. Phys. D* **22** (2013) 1350006.
16. L. D. Landau and E. M. Lifshitz, *Mechanics* (Butterworth- Heinemann, Oxford, UK, 1976).
17. A. E. Allahverdyan and Th. M. Nieuwenhuizen, *Phys. Rev. E* **75** (2007) 051124.
18. M. M. Verma and B. K. Yadav, arXiv: 1506.08649v2.